Pricing Power Perpetual Futures

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March 28, 2023

Abstract

This study discusses pricing for power perpetual futures. Perpetual futures have been first suggested by Shiller (1993) to help derivatives markets in illiquid assets. Whereas they rather failed to gain traction in traditional asset markets, perpetual futures have been heavily traded in cryptocurrencies where they are the main derivative instruments and beat spot markets in volume. Power perpetual futures have been recently introduced as leveraged version of perpetual futures. Traditionally people have assumed deterministic volatility in its pricing, and in this paper we introduce the pricing with stochastic volatility.

1 Introduction

Perpetual futures have been first suggested by Shiller (1993) to create derivatives markets in illiquid assets such as housing. Perpetual futures are cash settled and as the name suggests consist of a futures contract with no expiration date. The advantage of this is no rollovers and traders having to just pay the cash settled funding payments at frequent time intervals. They are straightforward to use with an easier user experience and provide a simpler solution for leveraged trading with auto liquidations. As discussed by White et al. (2021a) perpetual derivatives help with liquidity fragmentation and rolling over positions. Mainly in the case of smaller illiquid markets perpetual futures can offer several advantages over traditional futures contracts including greater flexibility, ease of use, and greater liquidity as they can be traded continuously without having to manage different expiry dates or rollovers. The main use cases of perpetual futures within traditional finance are in commodity and energy markets where they have gained popularity in recent years. Perpetual futures have been considerably more popular outside of traditional finance where they have been extensively used within cryptocurrency markets. Some estimates — such as by Carnegie Mellon University CyLab researchers [1] — suggest that volume of crypto derivatives exceeds spot trading by approximately a factor of five.

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Power perpetual futures —- which were initially introduced by Open and Paradigm Research [3] — represent a perpetual futures contract that is indexed to a power of the price index that the perp is tracking. White et al. [3] argues that power perpetual futures are endowed with positive convexity (or positive Gamma is option pricing language) and are thus more than a leveraged perpetual futures. In this construct convexity is introduced which creates an asymmetric payoff profile. Power perpetuals would thus trade at a premium of the index (otherwise there would be an arbitrage opportunity). Power perpetuals have different applications in crypto — such as the creation of delta neutral vaults and hedges for LPs in automated market makers — and are starting to surface in NFTs where HashCurve has been working on implementing power perpetual futures to replicate NFT exposures. Power perpetual futures are mainly interesting for ”options like” non-linear asymmetric payoff structures, and can help to generate liquid markets for these instruments as it concentrates liquidity better. In the case of options there are different strikes and maturities to handle which can scatter liquidity, and the exercise and roll-over can be practically difficult on-chain — this is especially important for non-major crypto assets with less active market making activity. White et al. [5] discuss everlasting options which work similarly to perpetual futures but in options which solves for the maturity fragmentation and need to roll over. However, it still requires to choose a strike price and thus concentrates liquidity in strikes which is, again, not ideal for lower liquidity crypto assets. Having protection over a fixed horizon is the main selling point of options, but if there is not enough liquidity to populate at different maturities and strikes market participants can be argued to be better off with concentrating the liquidity and offering nonlinear payoffs through power perpetual futures.

In this paper we expand the pricing model of power perpetual futures of White et al. [3] with a stochastic volatility component. The main disadvantage of the original constant volatility construct is that it assumes that the volatility deterministic and constant. More specifically, it does not account for the stochastic variability coming from natural volatility dynamics which are impacted by overall market sentiment and momentum. Especially in market with a limited range of traded derivatives — such as cryptocurrencies — there is a significant proportion of unspanned volatility. Furthermore, the former formulation does not account for the correlation (leverage effect) between the spot and volatility processes and subsequently differing sensitivities at different intra-period “moneyness”. While volatility is stochastic, assuming deterministic volatility would cause significant mispricing in power perpetuals of non-major crypto assets as the spot market of these assets is plagued by volatile behavior, volatility clustering, liquidity events, etc. In most use cases where perpetual futures are used there is no option surface to back out the implied volatility. In most cases (and especially during relatively low volatility periods such as during the time of publication of this paper) it would lead to an underestimation of the integrated variance. These assumptions will play less of a prominent role as market liquidity develops and the implied volatilities become directly observable. These considerations are especially important for perpetual futures in less developed or more illiquid markets and markets at ”inception” where there is limited historic data availability – i.e. considerations for which perpetual futures have been initially developed.

2 Derivation of Pricing Model for Power Perpetual Futures

In this section we discuss pricing of power perpetual futures. We start with the conventional pricing mechanism with deterministic volatility, and extend it with stochastic volatility. Following [3], the standard pricing formula for a power perpetual with power p is given as

$$V_p = S^p \cdot \frac{1}{2\exp(-f\frac{p-1}{2}(2r + pv^2)) - 1} \tag{1}$$

where S is the current spot price, r the risk-free rate, v the annualized volatility, and f the funding period in years.

The main disadvantage of this construct is that it assumes that the volatility deterministic and thus rather constant. More specifically, it does not account for the stochastic variability coming from unspanned volatility dynamics, which are impacted by overall market sentiment and momentum. Especially in market with a limited range of traded derivatives — such as cryptocurrencies — there is a significant proportion of unspanned volatility. Furthermore, the former formulation does not account for the correlation (leverage effect) between the spot and volatility processes and subsequently differing sensitivities at different intra-period "moneyness". Hence, in a departure from standard Black-Scholes assumptions, these additional sources of stochasticity must be accounted for, especially in a low liquidity environment. Note that especially at inception, markets do not have liquidity and some assumptions on price dynamics are required to guide pricing and price discovery until interpolation of vanilla market prices are possible. These assumptions will play less of a prominent role as market liquidity develops and the implied volatilities become directly observable. These considerations are especially important for perpetual futures in less developed markets such as crypto and NFTs – i.e. considerations for which perpetual futures have been initially developed.

Let the joint spot NFT price S_t and volatility v_t dynamics be given by (Schobel-Zhu model of [4]):

$$\begin{aligned} dS_t &= rS_t dt + v_t S_t dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma_v dW_t^v \end{aligned} \tag{2}$$

where W_t^S, W_t^v are standard Wiener processes with correlation coefficient $< dW_t^S, dW_t^v >= \rho$. κ controls the reversion speed whereas θ determines long-term average level. σ_v controls the volatility of volatility. For comparison, this reduces to the popular Heston model which introduces stochastic volatility to a Black-Scholes type options pricing model. More specifically, it uses Cox-Ingersoll-Ross (CIR) dynamics to the instantaneous variance process, under the following parameter restrictions:

$$\begin{aligned}\kappa &= \frac{1}{2}\kappa_h \\ \xi &= \frac{1}{2}\sigma_h \\ \theta &= 0 \\ \theta_h &= \frac{\sigma^2}{\kappa_h}.\end{aligned} \tag{3}$$

Under these assumptions, the expiring futures contract price F(t) is given as the following product

$$\mathbf{F}(t) = S^p \cdot D(t) \cdot E(t) \tag{4}$$

where the correlation adjusted discount factor D(t) is given by

$$D(t) = \exp\left(p\left(rt - \frac{\rho p}{2\sigma_v^2}\left(v^2 + v^2 t\right)\right)\right) \tag{5}$$

and the exponential transform factor E(t) representing the risk-neutral probability of (8) is given by and decomposes to

$$E(t) = A(t)\exp\left(B(t)v^2 + C(t)v\right) \tag{6}$$

with

$$\begin{aligned}A(t) = \frac{1}{\sqrt{\mathbf{G}(\gamma_1,\gamma_2)}} \cdot \exp\Bigg(&\frac{\kappa t}{2} + \frac{\kappa^2\theta^2\gamma_1^2 - \gamma_3^2}{2\xi^2\gamma_1^3}\left(\frac{\sinh(\gamma_1 t)}{\mathbf{G}(\gamma_1,\gamma_2)} - \gamma_1 t\right) \\ &+ \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\,\gamma_3}{\sigma_v^2\gamma_1^3}\left(\frac{\cosh(\gamma_1 t - 1)}{\mathbf{G}(\gamma_1,\gamma_2)}\right)\end{aligned} \tag{7}$$

$$B(t) = \frac{1}{2\sigma_v^2}\left(\kappa - \frac{\gamma_1\mathbf{H}(\gamma_1,\gamma_2)}{\mathbf{G}(\gamma_1,\gamma_2)}\right) \tag{8}$$

$$C(t) = \frac{1}{2\sigma_v^2}\left(\frac{\kappa\theta\gamma_1 - \gamma_2\gamma_3 + \gamma_3\mathbf{H}(\gamma_1,\gamma_2)}{\mathbf{G}(\gamma_1,\gamma_2)} - \kappa\theta\gamma_1\right) \tag{9}$$

Here the constant coefficients are

$$\begin{aligned}\gamma_1 &= \sqrt{\kappa^2 + 2\sigma_v^2 p\left(\frac{1 - p(1-\rho^2)}{2} - \frac{p\rho\kappa\theta}{\sigma_v}\right)} \\ \gamma_2 &= \frac{\kappa - 2\sigma_v^2\left(p\rho\kappa\theta\right)}{\sigma_v\gamma_1} \\ \gamma_3 &= \kappa^2\theta - \frac{\sigma_v p\rho}{2}\end{aligned} \tag{10}$$

and hyperbolic functions given by

$$\begin{aligned} \mathbf{G}(\gamma_1, \gamma_2) &= \cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) \\ \mathbf{H}(\gamma_1, \gamma_2) &= \sinh(\gamma_1 t) + \gamma_2 \cosh(\gamma_1 t) \end{aligned} \tag{11}$$

The pricing formula of (1) should then be modified to

$$V_p^v = \frac{S^p}{2\mathbf{F}(-f) - 1} \tag{12}$$

under the convergence region of

$$\mathbf{F}(f) < 2. \tag{13}$$

where we can see that the leverage effect parameter ρ and vol of vol σ_v play a prominent role in the formulation.

The derivation of the above result could be done by applying the Ito's formula to $f(x) = x^2$ and the dynamics given in (2) we have

$$dv_t^2 = 2\kappa \left(\frac{\sigma_v^2}{2\kappa} + \theta v_t - v_t^2 \right) dt + 2\sigma_v v_t dW_t. \tag{14}$$

The tower property gives us

$$\mathbf{E}[S_t^p] = S_t^p e^{prt} \mathbf{E} \left[\exp \left\{ \frac{p}{2} \left(p(1-\rho^2) - 1 \right) \int_0^t v_s^2 ds + p\rho \int_o^t v_s dWs \right\} \right] \tag{15}$$

$$\begin{aligned} \mathbf{E}[S_t^p] = S_t^p \exp \left\{ prt - \frac{p\rho}{2\sigma_v} \left(v^2 + \sigma_v^2 t \right) \right\} \cdot \\ \mathbf{E} \left[\exp \left\{ - \int_o^t \left((1 - p(1-\rho^2) v_s^2 + \left(\frac{p\rho\kappa\theta}{2} \right) v_s \right) ds + \left(\frac{\sigma_v p\rho}{2\sigma_v} \right) v_t^2 \right\} \right] \end{aligned} \tag{16}$$

The computation of the last risk-neutral transform term follows by rewriting the terms of the orthogonalized Wiener processes $\bar{W}_t = \rho W_t + \sqrt{1-\rho^2} W_t^{\perp}$ and then from the derivation in [4] of the closed forms of the transform function

$$f(\phi) = \mathbf{E} \left[\exp \left[-rt - v + (1 + i\phi) v_T \right] \right]. \tag{17}$$

It should be noted that by making the volatility component stochastic we would introduce procyclical volatility which would have a net increase in the convexity of the instrument.

Finally, the interest rates, while also stochastic, are directly observable and to be re-calibrated daily, hence remain treated as constant.

3 Recap: Pricing Model for Power Perpetual Futures

Under the Schobel-Zhu model [4] for stochastic volatility, the pricing formula of a power perpetual future contract with power p becomes

$$V_p^v = \frac{S^p}{2\mathbf{F}(-f) - 1} \tag{18}$$

where S stands for the spot price, f the funding period, and F(t) the expiring futures contract which is given by

$$\mathbf{F}(t) = S^p \cdot D(t) \cdot E(t) \tag{19}$$

with discount term D(t)

$$D(t) = \exp\left(p\left(rt - \frac{\rho p}{2\sigma_v^2}\left(v^2 + v^2 t\right)\right)\right) \tag{20}$$

and exponential transform term E(t)

$$E(t) = A(t)\exp\left(B(t)v^2 + C(t)v\right) \tag{21}$$

where

$$\begin{aligned} A(t) = \frac{1}{\sqrt{\mathbf{G}(\gamma_1, \gamma_2)}} \cdot \exp\Bigg(&\frac{\kappa t}{2} + \frac{\kappa^2\theta^2\gamma_1^2 - \gamma_3^2}{2\xi^2\gamma_1^3}\left(\frac{\sinh(\gamma_1 t)}{\mathbf{G}(\gamma_1, \gamma_2)} - \gamma_1 t\right) \\ &+ \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\,\gamma_3}{\sigma_v^2\gamma_1^3}\left(\frac{\cosh(\gamma_1 t - 1)}{\mathbf{G}(\gamma_1, \gamma_2)}\right)\Bigg) \end{aligned} \tag{22}$$

$$B(t) = \frac{1}{2\sigma_v^2}\left(\kappa - \frac{\gamma_1 \mathbf{H}(\gamma_1, \gamma_2)}{\mathbf{G}(\gamma_1, \gamma_2)}\right) \tag{23}$$

$$C(t) = \frac{1}{2\sigma_v^2}\left(\frac{\kappa\theta\gamma_1 - \gamma_2\gamma_3 + \gamma_3\mathbf{H}(\gamma_1, \gamma_2)}{\mathbf{G}(\gamma_1, \gamma_2)} - \kappa\theta\gamma_1\right) \tag{24}$$

with constant coefficients

$$\begin{aligned} \gamma_1 &= \sqrt{\kappa^2 + 2\sigma_v^2 p\left(\frac{1 - p(1-\rho^2)}{2} - \frac{p\rho\kappa\theta}{\sigma_v}\right)} \\ \gamma_2 &= \frac{\kappa - 2\sigma_v^2\left(p\rho\kappa\theta\right)}{\sigma_v\gamma_1} \\ \gamma_3 &= \kappa^2\theta - \frac{\sigma_v p\rho}{2} \end{aligned} \tag{25}$$

and hyperbolic functions

$$\begin{aligned} \mathbf{G}(\gamma_1, \gamma_2) &= \cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) \\ \mathbf{H}(\gamma_1, \gamma_2) &= \sinh(\gamma_1 t) + \gamma_2 \cosh(\gamma_1 t) \end{aligned}$$

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